

FASTER AND BETTER CIRCUIT ANALYSIS TECHNIQUES REQUIRE MINIMUM ALGEBRA

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The systematic and formal method of nodal or loop analysis is the universally adopted method of teaching network theory at the undergraduate level with rather little emphasis on network manipulation and transformation. Although the matrix algebra of formal network analysis can be easily and efficiently carried out *numerically* by a computer, it fails hopelessly when *analytical* answers are needed to help us acquire more insight into and familiarity with the operation of the circuit. Consequently, for most junior analog design engineers the only route to understanding and designing circuits better is mostly through the slow process of cut and try, either by breadboarding or using CAD tools, and not through analytical methods they learned in school. This problem has been recognized by Professor R.D. Middlebrook of Caltech who over the years as a consultant and an educator has developed efficient analytical methods which are revolutionary in deriving faster and better analytical answers while avoiding the horrors of runaway algebra^{1,2,3}. In this article I will illustrate how one of these analytical methods, known as the Extra Element Theorem² (EET), can cut through the algebra like a hot knife through butter.

The circuit, in Fig. 1 is taken directly from a well-known textbook by L. O. Chua and Pen-Min Lin in Computer Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques⁴ in which the input resistance R_{in} is to be determined as a function of the transconductance g_m using the parameter extraction method. Because the parameter extraction method requires considerable amount of matrix manipulations, which would become prohibitive if all elements were in symbolic form, the authors have assigned numerical values to R_1 through R_5 , as shown in Fig. 1, to obtain

$$R_{in} = \frac{96.3 + 5.1 g_m}{137.7 + 10.5 g_m} \quad (1a)$$

which can be rewritten in a somewhat better form by making the leading term in the numerator and the denominator unity

$$R_{in} = .7 \frac{1 + 19 \frac{5.3}{96.3} g_m}{1 + g_m \frac{10.5}{137.7}} \approx \frac{1}{2} \quad (1b)$$

One can see directly from the improved form of Eq. (1 b) that if g_m is zero, then $R_{in} = 70$. (Equation (1a) or (1b) is sometimes referred to as a *bilinear* function of g_m because the numerator and denominator are each a linear function of g_m .) In what follows, I would like to show how we can determine R_{in} entirely in symbolic form in a few *simple* steps using the Extra Element Theorem² (EET) twice in succession. This theorem allows us to remove elements from the circuit (impedance elements and dependent sources) in order to *analyze only a set of simpler circuits*. Here is the solution.

Solution

Clearly, the two elements that are giving us the most trouble in this circuit are g_m and R_B . So we first take out the dependent current source by letting $g_m = 0$ and obtain the circuit in Fig. 2 which is an ordinary bridge circuit with input resistance R'_{in} . In trying to determine R'_{in} , we see that R_B is the only element left which causes difficulty and so we take that one out too and obtain the circuit in Fig. 3a.

We now reinstate R_B and g_m by successive application of the EET. From Fig. 3a we can immediately write by inspection

$$R'_{in} = (R_1 + R_3) \parallel (R_2 + R_4) \quad (2)$$

All we need to do in order to determine R'_{in} are two more simple calculations as shown in Figs. 3b and 3c. In Fig. 3b, we determine the input resistance at port (B) with the input port short and obtain by inspection

$$\mathcal{R}^{(B)} = R_1 \parallel R_3 + R_2 \parallel R_4 \quad (3)$$

In Fig. 3c we determine the input resistance at port (B) with the input port open and obtain by inspection

$$R^{(B)} = (R_1 + R_2) \parallel (R_3 + R_4) \quad (4)$$

Now the EET tells us that R'_{in} is given by

$$R'_{in} = R'_{in} \Big|_{R_B \rightarrow \infty} \frac{1 + \frac{\mathcal{R}^{(B)}}{R_B}}{1 + \frac{R^{(B)}}{R_B}} \quad (5)$$

Hence, the input resistance to a bridge circuit is given by

$$R'_{in} = (R_1 + R_3) \parallel (R_2 + R_4) \frac{1 + \frac{R_1 \parallel R_3 + R_2 \parallel R_4}{R_B}}{1 + \frac{(R_1 + R_2) \parallel (R_3 + R_4)}{R_B}} \quad (6)$$

Equation (6) is a low-entropy expression¹ because it's bilinear form with respect to R_B is in terms of series and parallel combinations of the remaining resistances R_1 through R_4 which clearly shows the influence of R_B on R'_{in} .

In a second application of EET, we reinstate g_m by performing two additional simple calculations with the dependent current source g_m at port (m) replaced by an independent current source i_m pointing in the opposite direction of g_m as shown in Figs. 4a and 4b. In Fig. 4a, we determine the reverse gain v_1/i_m , (transresistance) with the input port short. Using the current division formula between R_B and $R_1 \parallel R_3 + R_2 \parallel R_4$ we obtain by inspection

$$\mathcal{A}^{(m)} = \left. \frac{v_1}{i_m} \right|_{(in) \rightarrow \text{short}} = \frac{R_B}{R_B + R_1 \parallel R_3 + R_2 \parallel R_4} R_1 \parallel R_3 \quad (7)$$

In Fig. 4b we repeat the same with the input port open. Using the current division formula between the branches $R_1 + R_2$ and $R_B \parallel (R_3 + R_4)$ we obtain by inspection

$$\mathcal{A}^{(m)} = \left. \frac{v_1}{i_m} \right|_{(in) \rightarrow \text{open}} = \frac{R_B \parallel (R_3 + R_4)}{R_B \parallel (R_3 + R_4) + R_1 + R_2} R_1 \quad (8)$$

The EET now tells us that R_{in} is given by

$$R_{in} = R_{in}|_{g_m \rightarrow 0} \frac{1 + g_m \mathcal{A}^{(m)}}{1 + g_m \mathcal{A}^{(m)}} \quad (9)$$

Substituting Eqs. (6), (7) and (8) in Eq. (9) we get the desired result

$$R_{in} = (R_1 + R_3 \parallel (R_2 + R_4)) \frac{1 + \frac{R_1 \parallel R_3 + R_2 \parallel R_4}{R_B}}{1 + \frac{(R_1 + R_2) \parallel (R_3 + R_4)}{R_B}} \frac{1 + g_m \frac{R_B R_1 \parallel R_3}{R_B + R_1 \parallel R_3 + R_2 \parallel R_4}}{1 + g_m \frac{R_1 R_B \parallel (R_3 + R_4)}{R_B \parallel (R_3 + R_4) + R_1 + R_2}} \quad (10)$$

Clearly the result in Eq. (10) is superior to the result in Eq. (1b) because it contains useful symbolic information about all the circuit elements. Not all symbolic expressions contain useful information unless the elements are grouped together in series and parallel combinations as in the low-entropy form of Eq. (10). If nodal or loop analysis was used, then the result would have come out in terms of a *single* numerator and a *single* denominator each containing the sum of products of four resistances at a time while the coefficient of g_m would contain the sum of products of five resistances at a time. Such a high -

entropy answer not only would have been prohibitively unpleasant to obtain, but would have contained absolutely no recognizable information about the operation of the circuit. This is precisely how engineers find out that nodal or loop analysis simply cannot help them understand the performance of a circuit even if they were to carry out the analysis and survive the algebra.

In a further example of how the EET can be invoked to obtain a simple low-entropy result, the bridge circuit in Fig. 1 can be made reactive by replacing R_B with a capacitor C_B as shown in Fig. 5. The purpose of this circuit is to illustrate the following three important points. First, a reactive circuit can be analyzed with the same ease as a resistive circuit without ever having to deal with the reactive impedance term $1/sC_B$. Second, single extraction of the parameter g_m is not useful because g_m affects the low-frequency asymptote as well as the pole and the zero of $Z_{in}(s)$. Third, by using the EET, $Z_{in}(s)$ can automatically be determined in factored pole-zero form.

Solution

First, we take out the capacitive reactance $Z_B = 1/sC_B$ and the dependent current source g_m and obtain the same circuit, as in Fig. 3a for which R'_{in} is given by Eq. (2). Next, reinstate the dependent current source g_m and determine $\mathcal{A}^{(m)}$ and $\mathcal{A}^{(m)}$ shown in Fig. 6 which are the same as before but with $R_B \rightarrow \infty$. Hence, using EET we obtain for R'_{in}

$$R'_{in} = (R_1 + R_3) \parallel (R_2 + R_4) - \frac{1 + g_m R_1 \parallel R_4}{1 + g_m \frac{R_1(R_3 + R_4)}{(R_3 + R_4) + R_1 + R_2}} \quad (11)$$

Finally, we reinstate the capacitive reactance and determine $\mathcal{R}^{(B)}$ and $\mathcal{R}^{(B)}$ shown in Fig. 7a and 7b. In Fig. 7a we have

$$i_T = \frac{v_T}{R_1 \parallel R_3 + R_2 \parallel R_4} + g_m \frac{v_T R_1 \parallel R_3}{R_1 \parallel R_3 + R_2 \parallel R_4} \quad (12)$$

so that

$$\mathcal{R}^{(B)} = \frac{R_1 \parallel R_3 + R_2 \parallel R_4}{1 + g_m R_1 \parallel R_3} \quad (13)$$

In light of Eq. (13) we have from Fig. 7b

$$\mathcal{R}^{(B)} = (R_3 + R_4) \parallel \frac{R_1 + R_2}{1 + g_m R_1} \quad (14)$$

Using the EET, we obtain for $Z_{in}(s)$

$$Z_{in}(s) = R'_{in} \frac{1 + sC_B \mathcal{R}^{(B)}}{1 + sC_B \mathcal{R}^{(B)}} \left[\frac{1 + s/s_z}{1 + s/s_p} \right] \quad (15)$$

where $R_o = R'_{in}$ is the low-frequency asymptote and

$$s_z = \frac{1}{C_B s_{\mathcal{R}}(B)} = \frac{1 + g_m R_1 \parallel R_3}{C_B (R_1 \parallel R_3 + R_2 \parallel R_4)} \quad (16)$$

$$s_p = \frac{1}{C_B R(B)} = \frac{1}{C_B (R_3 + R_4) \parallel \frac{R_1 + R_2}{1 + g_m R_1}} \quad (17)$$

The elegance and simplicity of this derivation illustrate how, when the extra element is a reactance in an otherwise resistive circuit, the pole and zero appear immediately in the low-entropy form of the EET correction factor.

The EET can also be applied to the determination of any kind of transfer function such as voltage gain, current gain or loop gain and can be extended to two or more extra elements^{3,5}.

References

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- [4] Leon O. Chua and Pen-Min Lin, *Computer Aided Analysis of Electronic Circuits: Algorithms and Computational Techniques* pp 568-569. Prentice Hall
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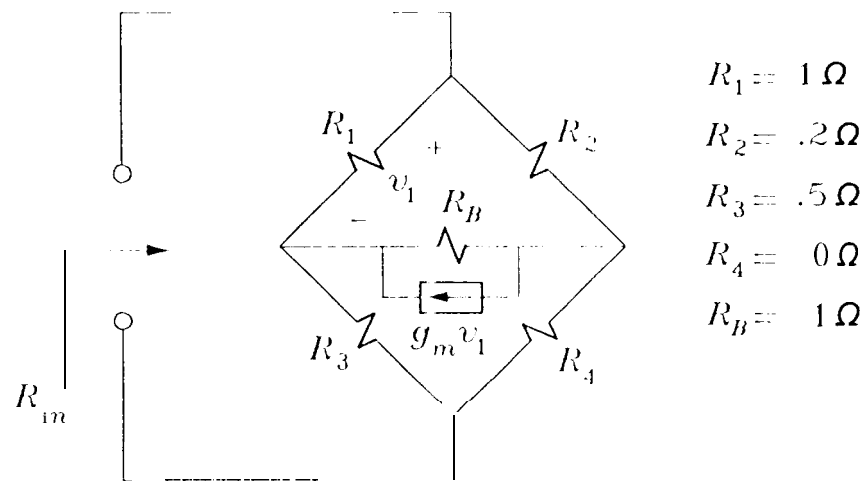


Figure 1

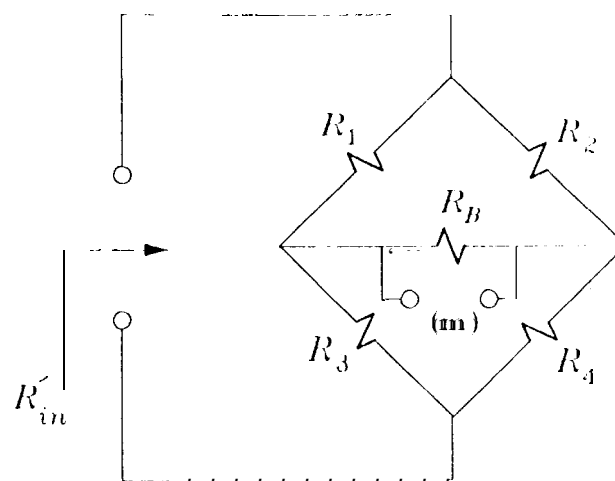


Figure 2

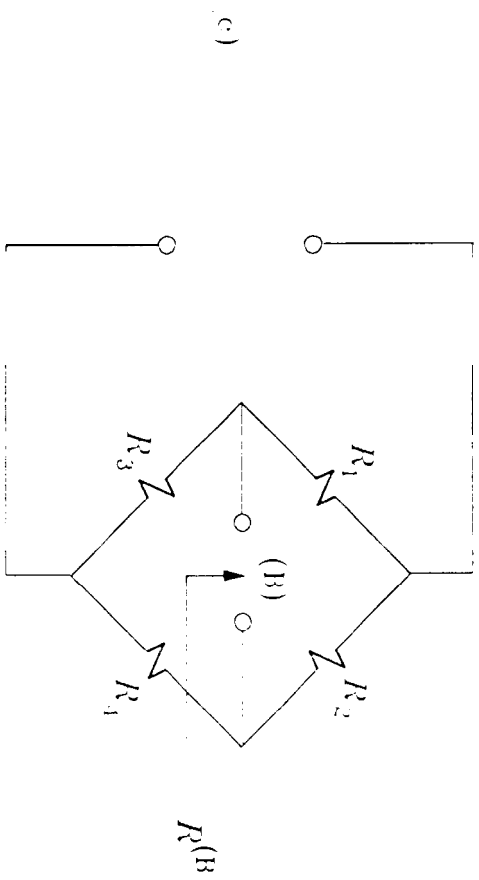
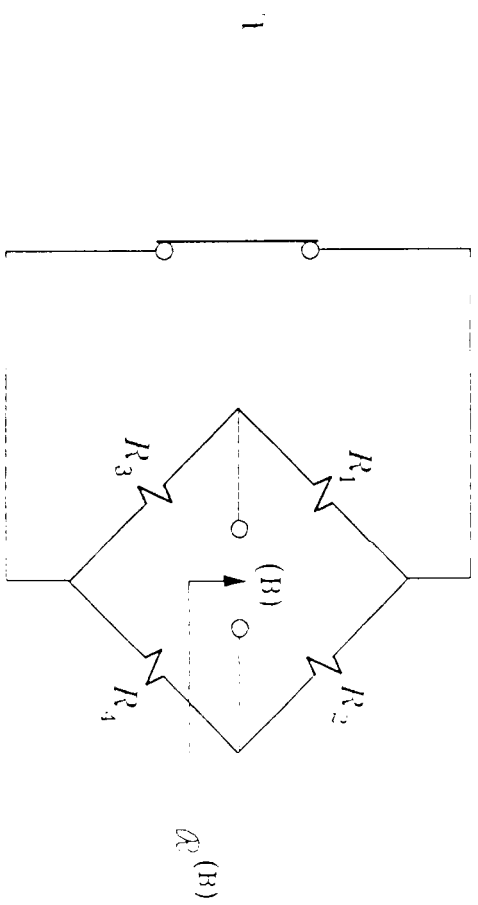
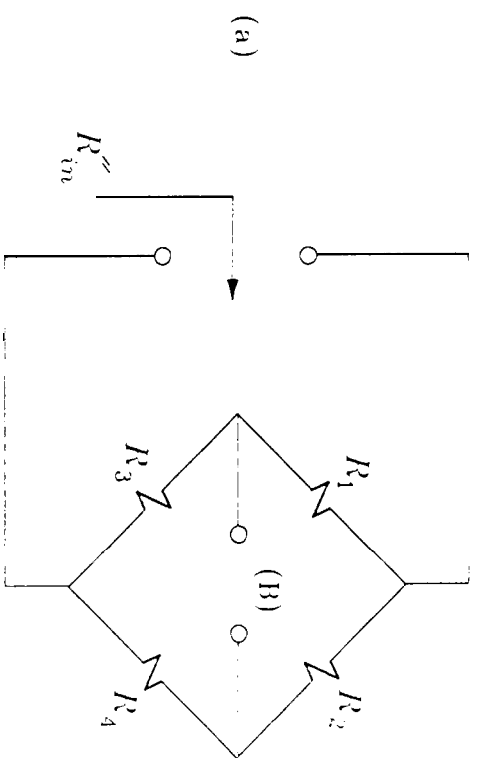


Figure 3

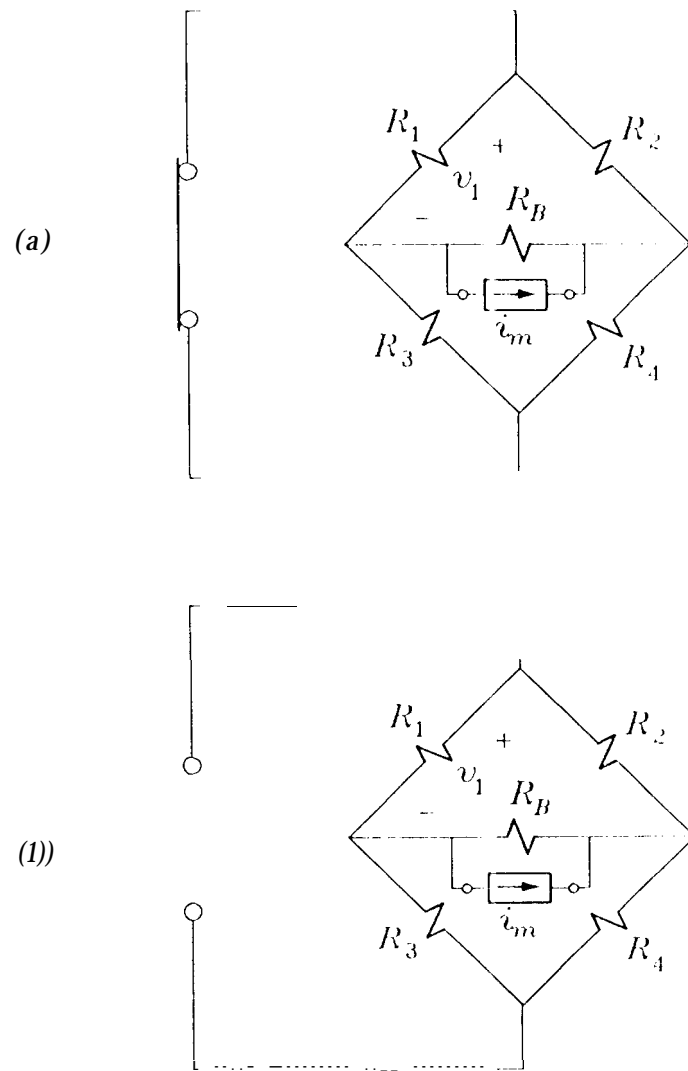


Figure 4

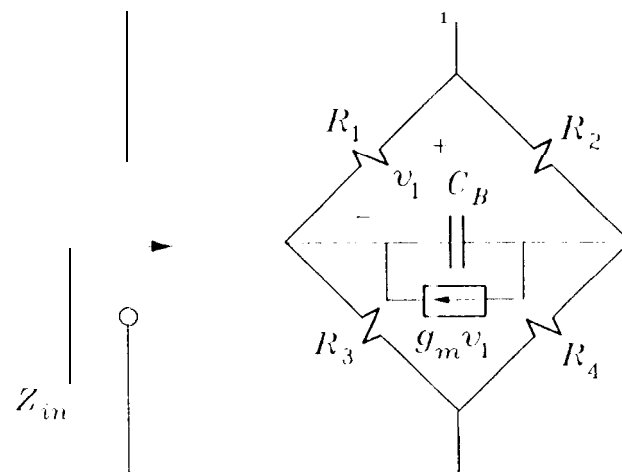


Figure 5

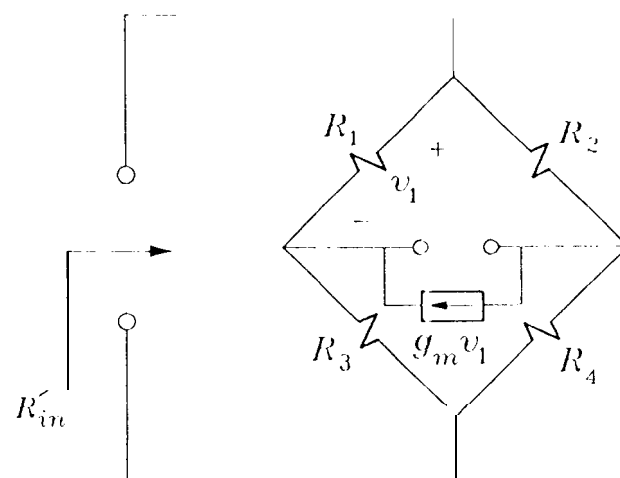


Figure 6

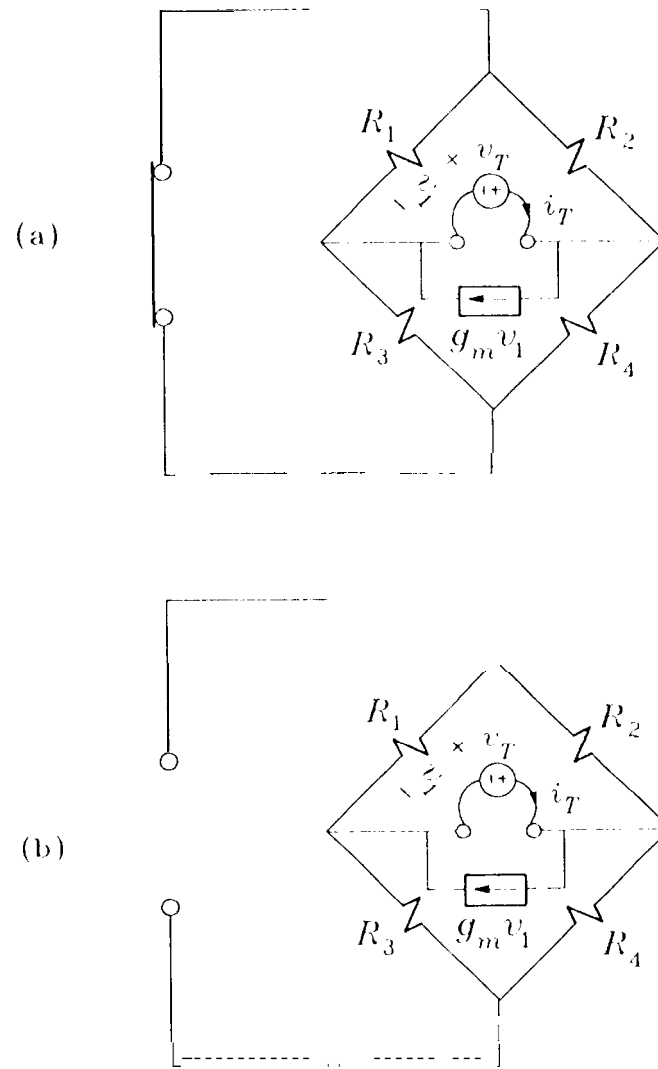


Figure 7